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On limit functions of minimizers of variational problems.

Toshihiro Ishibashi(石橋利裕)
Saitama University(埼玉大理)

概 要

We consider the uniqueness of viscosity solutions of a certain discontinuous partial differential equation (PDE for short) under the homogeneous Dirichlet boundary condition. This discontinuous PDE is derived from variational problems containning a certain L^p norm. We also show that the unique solution coincides with the distance function from the boundary.

1. Introduction

At first, we consider the following variational problem: Minimize the functional

$$G(w, Dw) := \frac{1}{p}\|Dw\|^p - \int_\Omega f w dx$$

subject to $w \in W_0^{1,p}(\Omega)$, where $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$, $f \in C(\bar{\Omega})$ is positive in Ω and $\|\cdot\|$ denotes various $L^p(\Omega, R^n)$ norms which are equivalent to the standard one. Moreover, we derive the limit PDE which the limit function of minimizers, as $p \to \infty$, satisfies.

In order to deal with the perfectly plastic tortion model, T. Bhattacharya - E. DiBenedetto - J. Manfredi [1] and B. Kawohl[6] considered the limit function of minimizers, as $p \to \infty$, of the following variational problem: Minimize the functional $G_2(w, Dw)$ subject to $w \in W_0^{1,p}(\Omega)$, where

$$G_2(w, Dw) := \frac{1}{p}\int_\Omega (|Dw|_2)^p dx - \int_\Omega f w dx. \tag{1.1$_p$}$$

Then, they showed that

$$\lim_{p\to\infty} u_p(x) = d_2(x) := \inf_{y\in\partial\Omega} |x-y|_2$$

uniformly $x \in \bar{\Omega}$, where we denote by $|\cdot|_2$ the Euclidean norm.

On the other hand, initiated by Aronsson's works, R. Jensen in [5] characterized absolute minimal Lipschitz extensions as unique viscosity solutions of

$$-\Delta_\infty w := -\sum_{i,j=1,\dots,n} w_{x_i} w_{x_j} w_{x_i x_j} = 0 \quad \text{in} \quad \Omega$$

under the Dirichelet boundary condition. To show the uniqueness of this equation, Jensen introduced the following PDE:

$$\min\{|Dw|_2 - \epsilon, -\Delta_\infty w\} = 0 \quad \text{in} \quad \Omega, \tag{1.2}$$

which the limit function of minimizers of $(1.1)_p$ with $f = \epsilon^p$ $(\epsilon > 0)$ satisfies, and obtained the uniqueness result for solutions of (1.2) under the Dirichlet boundary condition.

In the previous work with S. Koike [3] we considered variational problems which contain several $L^p(\Omega, R^n)$ norms equivalent to the standard one. As a typical case, we considered the following variational problem: Minimize the functional $G_p(w, Dw)$ subject to $w \in W_0^{1,p}(\Omega)$, where

$$G_p(w, Dw) := \frac{1}{p}\int_\Omega (|Dw|_p)^p dx - \int_\Omega fw dx. \tag*{$(1.3)_p$}$$

Here and later, for $\xi = (\xi_1, \dots, \xi_n)$ and $\alpha \in [1, \infty)$, we define the norm $|\xi|_\alpha$ of R^n by

$$|\xi|_\alpha := \left(\sum_{i=1,\dots,n} |\xi_i|^\alpha\right)^{\frac{1}{\alpha}}.$$

Then, we showed that the limit function of minimizers of $(1.3)_p$, as $p \to \infty$, satisfies the limit PDE:

$$\min\{|Dw|_\infty - 1, F_\infty(Dw, D^2 w)\} = 0 \quad \text{in} \quad \Omega \tag{1.4}$$

in the viscosity sense, where, for all $\xi = (\xi_1, \dots, \xi_n) \in R^n$ and $X = (X_{ij}) \in S^n$, we define $F_\infty(\xi, X)$, $|\xi|_\infty$ and $I[\xi]$ in the following way:

$$F_\infty(\xi, X) := -\sum_{i \in I[\xi]} X_{ii}, \quad |\xi|_\infty := \max_{i \in \{1,\dots,n\}} |\xi_i|$$

$$\text{and} \qquad I[\xi] := \{i \in \{1, \dots, n\}; |\xi|_\infty = |\xi_i|\}.$$

Here, S^n denotes the set of $n \times n$ symmetric matrices equipped with the standard order. More precisely, we obtain

Theorem 1.1 (Theorem 3.1 in [3]). *Let $u_p \in W^{1,p}(\Omega)$ be the minimizer of variational problem $(1.3)_p$. Then, there exists a subsequence $\{u_{p_j}\}_{j\in N}$ and a function $u \in W^{1,\infty}(\Omega)$ such that $u_{p_j} \to u$ uniformly in $\bar{\Omega}$ as $p_j \to \infty$ and u satisfies* (1.4) *in the viscosity sense.*

Now, we recall the definition of viscosity solutions.
Definition. For a given $F : R^n \times S^n \to R$, we call $u \in C(\Omega)$ a viscosity supersolution (resp., subsolution) of

$$F(Dw, D^2w) = 0 \qquad \text{in} \quad \Omega \tag{1.5}$$

if

$$F^*(\xi, X) \geq 0 \quad (\text{resp.}, F_*(\xi, X) \leq 0)$$

for all $x \in \Omega$ and $(\xi, X) \in \bar{J}^{2,-}u(x)$ (resp., $(\xi, X) \in \bar{J}^{2,+}u(x)$), where F_* and F^* are lower and upper semicontinuous envelopes of F, respectively, and

$$J^{2,-}u(x) := \left\{ (\xi, X) \in R^n \times S^n; \quad \begin{array}{rl} u(y) \geq & u(x) + \langle \xi, y - x \rangle \\ & +\frac{1}{2}\langle X(y-x), y-x \rangle \\ & +o(|x-y|_2^2) \quad \text{as} \quad y \to x \end{array} \right\},$$

$$J^{2,+}u(x) := \left\{ (\xi, X) \in R^n \times S^n; \quad \begin{array}{rl} u(y) \leq & u(x) + \langle \xi, y - x \rangle \\ & +\frac{1}{2}\langle X(y-x), y-x \rangle \\ & +o(|x-y|_2^2) \quad \text{as} \quad y \to x \end{array} \right\},$$

and $\bar{J}^{2,\pm}u(x)$ is the graph closure of $J^{2,\pm}u(x)$, i.e.,

$$\bar{J}^{2,\pm}u(x) := \left\{ (q, X) \in R^n \times S^n \ ; \quad \begin{array}{c} \exists x^m \in \Omega \text{ and } \exists (q^m, X^m) \in J^{2,\pm}u(x^m) \\ \text{such that } \lim_{m\to\infty} (x^m, u(x^m), q^m, X^m) \\ = (x, u(x), q, X) \end{array} \right\}$$

We call $u \in C(\Omega)$ a viscosity solution of (1.5) if it is a viscosity supersolution and a viscosity subsolution of (1.5).

Remark. We note that if $F^*(\xi, X) \geq 0$ (resp., $F_*(\xi, X) \leq 0$) for all $x \in \Omega$ and $(\xi, X) \in J^{2,-}u(x)$ (resp., $(\xi, X) \in J^{2,+}u(x)$), then u is a viscosity supersolution (resp., subsolution) of (1.5).

Our interest is to obtain the uniqueness result for viscosity solutions of (1.4) under the Dirichlet boundary condition. Moreover, we are interested in getting the formula of the limit function. To establish the unequeness result, we will show the comparison principle between viscosity supersolutions and viscosity subsolutions. However, we notice that F_∞ has a serious discontinuity with respect to ξ variables, which causes a difficulty to show the comparison principle. Indeed, since there is a gap between $F_{\infty *}$ and F_∞^*, even for a classical supersolution v and a classical subsolution u of (1.4), we cannot avoid the case when $F_{\infty *}(Du(x^0), D^2u(x^0)) < F_\infty^*(Dv(x^0), D^2v(x^0))$ in general, when $u - v$ attains a local maximum at x^0. Thus, to our knowledge, we cannot apply the standard argument to show the comparison principle. In the previous work [3], to overcome this difficulty, we imposed the local concavity of viscosity supersolutions (or the local convexity of viscosity subsolutions) for our comparison result. On the other hand, we note that we obtained the formula for the limit function in [3], i.e.,

$$\lim_{p\to\infty} u_p(x) = d_1(x) := \inf_{y\in\partial\Omega} |x - y|_1 \tag{1.6}$$

uniformly $x \in \bar{\Omega}$ when Ω is convex.

Here, we show the uniqueness of solutions of (1.4) under the homogeneous Dirichlet boundary condition without this concavity assumption and, moreover, prove (1.6) for general Ω. To this end, we compare viscosity supersolutions with the expected solution d_1 and viscosity subsolutions with d_1, respectively. In the next section, we present some properties of d_1, and using these properties, we show the uniqueness result for viscosity solutions of (1.4) under the homogeneous Dirichlet boundary condition.

In order to explain the reason why our equivalent norm in $(1.3)_p$ could be an exremal one, we consider the variational problem: For fixed $\alpha \in [2, \infty)$, minimize the functional

$$G_\alpha(w, Dw) := \frac{1}{p}\int_\Omega (|Dw|_\alpha)^p dx - \int_\Omega fw dx \qquad (1.7)_p$$

subject to $w \in W_0^{1,p}(\Omega)$. Using the standard argument, we can derive the following limit PDE which the limit function of minimizers satisfies in the viscosity sense:

$$\min\{|Du|_\alpha - 1, F_\alpha(Du, D^2u)\} = 0 \quad \text{in } \Omega. \tag{1.8}$$

Here, for all $\xi = (\xi_1, ..., \xi_n) \in R^n$ and $X = (X_{ij}) \in S^n$, we define $F_\alpha(\xi, X)$ in the following manner:

$$F_\alpha(\xi, X) := -\sum_{i,j=1}^{n} |\xi_i|^{\alpha-2}\xi_i X_{ij}\xi_j|\xi_j|^{\alpha-2}.$$

Since F_α is continuous in (ξ, X), using the Jensen's argument in [5], we can show the comparison principle for viscosity solutions of (1.8). Moreover, we obtain the

$$\lim_{p\to\infty} u_p(x) = d_{\alpha^*}(x) := \inf_{y\in\partial\Omega} |x-y|_{\alpha^*}$$

uniformly $x \in \bar{\Omega}$ because it is easy to show that d_{α^*} is a viscosity solution of (1.8). Here u_p is the minimizer of $(1.7)_p$ and α^* is the Hölder conjugate of α, i.e.,

$$\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1.$$

By taking into account of this observation, our choice of the equivalent norm in $(1.3)_p$ can be interpreted as the case when $\alpha = \infty$.

2. Uniqueness.

In this section, we consider the uniqueness result. To obtain the uniqueness result, we present some results for the property of d_1. Actually, we obtain following properties.

Lemma 2.1. *d_1 is a viscosity solution of*

$$|Dw|_\infty - 1 = 0 \qquad in \quad \Omega. \tag{2.1}$$

Lemma 2.2. *For all $x^0 \in \Omega$ and $(\xi, X) \in \bar{J}^{2,-}d_1(x^0)$, if $i \in I[\xi]$, then $X_{ii} \leq 0$. In particular,*

$$F_{\infty *}(\xi, X) \geq 0.$$

Remark. As a consequence of these properties, we can see that d_1 is a viscosity solution of (1.4).

In the rest of this section, we show the comparison principle to obtain the uniqueness result for viscosity solutions of (1.4). However, since $F_\infty(\xi, X)$ contains a serious discontinuity with respect to ξ variables, to our knowledge, we cannot apply the standard argument to prove the comparison principle for (1.4). To overcome this difficulty, we separately compare viscosity supersolutions with d_1 and viscosity subsolutions with d_1. To this end, it will turn out that fine properties of d_1 are useful.

First, we recall a construction of strictly viscosity subsolutions (originally by [5]) which approximate viscosity subsolutions.

Lemma 2.3 (Lemma 4.2 in [3]). *Let $u \in C(\bar{\Omega})$ be a viscosity subsolution of* (1.4). *For any $\epsilon > 0$, there are a function $\bar{u}$ and a constant $\tau > 0$ satisfying the following properties:*

$$\max_{\bar{\Omega}} |u - \bar{u}|_2 < \epsilon.$$

$\bar{u}$ is a viscosity subsolution of

$$\min\{|Dw|_\infty - 1, F_\infty(Dw, D^2w)\} + \tau = 0 \qquad in \quad \Omega.$$

Using these lemmas, we obtain the comparison principle for viscosity solutions of (1.4).

Theorem 2.4. *Let $u \in C(\bar{\Omega})$ be a viscosity subsolution of* (1.4) *and $v \in C(\bar{\Omega})$ be a viscosity supersolution of* (1.4). *If we assume that $\sup_{\partial\Omega} u \le \inf_{\partial\Omega} v$, then we have $u \le v$ in Ω.*

Proof. Without loss of generality, we may assume that

$$\sup_{\partial\Omega} u \le 0 \le \inf_{\partial\Omega} v.$$

First, we shall compare v with d_1. Our purpose is to show that $d_1 \le v$ in Ω. Here, we note that v is a viscosity supersolution of (2.1). Thus, applying to the comparison result for viscosity solutions of eikonal equations (see [4]), we obtain that $d_1 \le v$ in Ω.

Next, we shall compare u with d_1. We argue by contradiction. We assume that $\max_{\partial\Omega}(u - d_1) < \max_{\bar{\Omega}}(u - d_1)$. By Lemma 2.3, there exist $\tau > 0$ and a function $\bar{u}$ satisfying that $\bar{u}$ is a viscosity subsolution of

$$\min\{|Dw|_\infty - 1, F_\infty(Dw, D^2w)\} + \tau = 0 \qquad \text{in} \quad \Omega,$$

and

$$\max_{\partial\Omega}(\bar{u} - d_1) < \max_{\bar{\Omega}}(\bar{u} - d_1). \tag{2.2}$$

At a maximum point $x_0 \in \Omega$, the gradient of $\bar{u}$ and d_1 are equal at least formally; $D\bar{u}(x_0) = Dd_1(x_0)$. Moreover, we obtain

$$D^2\bar{u}(x_0) \le D^2 d_1(x_0).$$

Since d_1 is a supersolution of (1.4), we have

$$|Dd_1(x_0)|_\infty - 1 \ge 0. \tag{2.3}$$

On the other hand, since $\bar{u}$ solves

$$\min\{|D\bar{u}|_\infty - 1, F_{\infty *}(D\bar{u}, D^2\bar{u})\} + \tau \leq 0 \qquad \text{in} \quad \Omega,$$

we obtain that

$$F_{\infty *}(D\bar{u}(x_0)), D^2\bar{u}(x_0)) + \tau \leq 0. \tag{2.4}$$

This is a contradiction to Lemma 2.2. Indeed, by (2.4) and Lemma 2.2, we have

$$-\tau \geq F_{\infty *}(D\bar{u}(x_0)), D^2\bar{u}(x_0)) \geq F_{\infty *}(Dd_1(x_0), D^2 d_1(x_0)) \geq 0.$$

By using the standard argument of the theory of viscosity solutions, this formal argument can be justified. Thus, we get $d_1 \geq u$ in Ω. □

As a consequence of Theorem 2.4, we can easily show the uniqueness result for viscosity solutions of (1.4) under the homogeneous Dirichelet boundary condition. Moreover, we obtain the full sequence convergence of minimizers of $(1.3)_p$ and the formula (1.6) for the limit function.

Corollary 2.5. *If $u \in C(\bar{\Omega})$ is a viscosity solution of* (1.4) *and $u \equiv 0$ on $\partial\Omega$, then $u = d_1$ in $\bar{\Omega}$.*

Corollary 2.6. *Let u_p be the minimizer of variational problem $(1.3)_p$. Then, we obtain that*

$$u_p \to d_1 \qquad \text{as} \quad p \to \infty$$

uniformly in $\bar{\Omega}$.

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